Correction Model Final Exam
Linear Algebra 2, April 2019

1 a) A complex inner product on V is a function $\langle \, , \, \rangle$ that assigns to each pair of vectors $x, y \in V$ a complex number $\langle x, y \rangle$ such that

1. $\langle x, x \rangle \geq 0$ for all $x \in V$, and $\langle x, x \rangle = 0$ if and only if $x = 0$.

2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in V$

3. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for all $\alpha, \beta \in \mathbb{C}$ and $x, y, z \in V$

Alternative definition: it is a function $\langle \, , \, \rangle$ from $V \times V$ to $\mathbb{C}$ such that .....

b) We check the 3 conditions:

1. Let $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{C}^n$. Then

$$\langle x, x \rangle = x^H x = \bar{x}_1 x_1 + \bar{x}_2 x_2 + \dots + \bar{x}_n x_n$$

$$= |x_1|^2 + |x_2|^2 + \dots + |x_n|^2 \geq 0$$

$= |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 \geq 0$

Also $\langle x, x \rangle = 0 \iff |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 = 0$
$\iff |x_i|^2 = 0$ for all i $\iff x_i = 0$ for all i
$\iff x = 0$

2. Let $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \; y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{C}^n.$

Then $\langle x, y \rangle = y^H x = \bar{y}_1 x_1 + \bar{y}_2 x_2 + \cdots + \bar{y}_n x_n$

Hence $\overline{\langle y, x \rangle} = \overline{\bar{x}_1 y_1 + \bar{x}_2 y_2 + \cdots + \bar{x}_n y_n}$

$= \bar{\bar{x}}_1 \bar{y}_1 + \bar{\bar{x}}_2 \bar{y}_2 + \cdots + \bar{\bar{x}}_n \bar{y}_n$

$= \bar{y}_1 x_1 + \bar{y}_2 x_2 + \cdots + \bar{y}_n x_n = y^H x = \langle x, y \rangle$

Alternatively: $\overline{\langle y, x \rangle} = \overline{x^H y} = (x^H y)^H$
$= y^H (x^H)^H = y^H x = \langle x, y \rangle$

3. Let $\alpha, \beta \in \mathbb{C}$, $x, y, z \in \mathbb{C}^n$

$\langle \alpha x + \beta y, z \rangle = z^H (\alpha x + \beta y)$

$= \alpha z^H x + \beta z^H y = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$

c. $A = A^H$. Let $x \in \mathbb{C}^n$

$$\langle x, Ax\rangle = (Ax)^H x = x^H A^H x$$
$$= x^H A x$$

On the other hand:

$$\overline{\langle x, Ax\rangle} = \overline{(Ax)^H x} = \overline{x^H A^H x}$$
$$= (x^H A^H x)^H = x^H (A^H)^H (x^H)^H$$
$$= x^H A x$$

Hence $\overline{\langle x, Ax\rangle} = \langle x, Ax\rangle$ so it must be real.

2. $A = \begin{pmatrix} 9 & -6 \\ 5 & -3 \end{pmatrix}$. We compute first the characteristic polynomial:

$$\det(A - zI) = \det\begin{pmatrix} 9-z & -6 \\ 5 & -3-z \end{pmatrix} =$$

$$(9-z)(-3-z) + 30 = z^2 - 6z + 3$$

By Cayley-Hamilton: $A^2 = 6A - 3I$.

Hence $A^3 = 6A^2 - 3A$

$= 6(6A - 3I) - 3A$

$= 33A - 18I$

Thus $A^4 = 33A^2 - 18A$

$= 33(6A - 3I) - 18A$

$= 180A - 99I$

Finally, $A^5 = 180A^2 - 99A$

$= 180(6A - 3I) - 99A$

$= 981A - 540I$

Therefore $A^5 - 981A = -540I$

and $-\frac{1}{540}A^5 + \frac{981}{540}A = I$

Take $\boxed{\alpha = -\frac{1}{540}, \quad \beta = \frac{981}{540}.}$

3. (a) Let λ be an eigenvalue and $x \neq 0$ such that $Ax = \lambda x$.

We have

$$x^T A x = x^T \lambda x = \lambda x^T x = \lambda \|x\|^2$$

Since $x^T A x > 0$ (positive definite!) we obtain that $\lambda \|x\|^2 > 0$. Since $\|x\|^2 > 0$ this implies that λ is real and $\lambda > 0$.

(b) To prove: λ_i eigenvalue of $A \Leftrightarrow$ λ_i^2 eigenvalue of A^2

$\boxed{\text{The easiest way to prove this is as follows:}}$

A is symmetric, so there exists an orthogonal matrix Q s.t. $A = Q^T \Lambda Q$ with
$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots & \\ & & & \lambda_n \end{pmatrix}$. This implies

$$A^2 = Q^T \Lambda Q Q^T \Lambda Q = Q^T \Lambda^2 Q.$$

Note that $\Lambda^2 = \begin{pmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^2 \end{pmatrix}$ and its diagonal elements are the eigenvalues of A^2.

Alternative (partial) proof is:
Let λ be an eigenvalue of A. There exists $x \neq 0$ such that $Ax = \lambda x$. Hence $A^2 x = \lambda A x = \lambda^2 x$, so λ^2 is an eigenvalue of A^2.
It then remains to prove that if μ is an eigenvalue of A^2 it must be of the form $\mu = \lambda^2$ with λ an eigenvalue of A. This is more difficult, so I prefer the first proof.

If somebody gives this partial proof he/she gets 3 points

(c) The singular values of A are obtained by taking the square roots of the eigenvalues of $A^T A$.

Since $A^T = A$ we have $A^T A = A^2$.

Hence, by (b), the singular values are $\sqrt{\lambda_1^2}, \dots, \sqrt{\lambda_n^2}$ so $\lambda_1, \dots \lambda_n$.

(d) Let $Q^T A Q = \Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$

and Q orthogonal. By permuting the columns of Q we can renumber the λ_i's so that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$.

Assume this has already been done.

Then we have

$$A = Q \Lambda Q^T$$

Define $U = Q$, $V = Q$. Then

$$A = U \Lambda V^T$$

with $\Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$

is a singular value decomposition

Note: subtract 2 points if the issue of permutation is not addressed

4. $M = \begin{pmatrix} 3 & 3 \\ -\sqrt{2} & \sqrt{2} \\ 3 & 3 \end{pmatrix}$

(a) $M^T M = \begin{pmatrix} 20 & 16 \\ 16 & 20 \end{pmatrix} = 4 \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$

$(5-z)^2 - 16 = z^2 - 10z + 9$

$= (z-1)(z-9)$

roots are $z = 1$ and $z = 9$. Thus the eigenvalues of $M^T M$ are 36 and 4

Singular values of M: $\sigma_1 = 6$, $\sigma_2 = 2$

(b) Find an orthogonal matrix V such that $M^T M V = V \Sigma^2$. Put $V = (v \ w)$

Then $M^T M v = 36 v \iff$

$$\begin{pmatrix} 20 & 16 \\ 16 & 20 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 36 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\Longleftrightarrow \quad \begin{aligned} 20v_1 + 16v_2 &= 36v_1 \\ 16v_1 + 20v_2 &= 36v_2 \end{aligned}$$

$$\Longleftrightarrow \quad 16v_2 = 16v_1 \qquad \text{Take } v = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Next

$$\begin{pmatrix} 20 & 16 \\ 16 & 20 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 4 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$\Longleftrightarrow \quad \begin{aligned} 20w_1 + 16w_2 &= 4w_1 \\ 16w_1 + 20w_2 &= 4w_2 \end{aligned}$$

$$\Longleftrightarrow \quad 16w_2 = -16w_1$$

$$\text{Take } w = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Thus : $\boxed{V = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}}$.

Next, we want

$$M = U \Sigma V^T \; ; \quad \boxed{\Sigma = \begin{pmatrix} 6 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}}$$

So

$$MV = U\Sigma \quad (*)$$

Besides, U has to 3×3 orthogonal.

Set $U = (u_1 \ u_2 \ u_3)$. From (*) we get

$$u_1 = \frac{1}{6} M v_1 \ ; \quad u_2 = \frac{1}{2} M v_2$$

Hence

$$u_1 = \frac{1}{6}\begin{pmatrix} 3 & 3 \\ -\sqrt{2} & \sqrt{2} \\ 3 & 3 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{6\sqrt{2}}\begin{pmatrix} 6 \\ 0 \\ 6 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$u_2 = \frac{1}{2}\begin{pmatrix} 3 & 3 \\ -\sqrt{2} & \sqrt{2} \\ 3 & 3 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 \\ -2\sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

We take $u_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and obtain the orthogonal matrix

$$\boxed{U = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}}$$

This yields $M = U \Sigma V^T$

(c) Best rank 1 approximation is

$$\hat{M} := U \begin{pmatrix} 6 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} V^T$$

$$= 6 \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= 6 \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \boxed{\begin{pmatrix} 3 & 3 \\ 0 & 0 \\ 3 & 3 \end{pmatrix}}$$

5. (a) The minimal polynomial of A is the monic polynomial $p(z)$ of smallest degree with the property that $p(A) = 0$

(b) Let $p(z) = p_n z^n + p_{n-1} z^{n-1} + \dots + p_1 z + p_0$

Let λ be an eigenvalue of A and

$x \neq 0$ be such that $Ax = \lambda x$

Then $A^k x = \lambda^k x$ for all $k = 0, 1, 2 \ldots$

Hence

$$p(A)x = (p_n A^n + p_{n-1} A^{n-1} + \cdots + p_1 A + p_0 I)x$$

$$= p_n A^n x + p_{n-1} A^{n-1} x + \cdots + p_1 A x + p_0 x$$

$$= (p_n \lambda^n + p_{n-1} \lambda^{n-1} + \cdots + p_1 \lambda + p_0) x$$

$$= p(\lambda) x.$$

(c) Let λ be an eigenvalue of A. Then there exists $x \neq 0$ with $Ax = \lambda x$

We then have

$$0 = p_{min}(A)x = p_{min}(\lambda)x.$$

Since $x \neq 0$ this implies $p_{min}(\lambda) = 0$

6.
$$A = \begin{pmatrix} 0 & 0 & -1 \\ -2 & 1 & -1 \\ 1 & 0 & 2 \end{pmatrix}$$

(a.)

$$\det(A - zI) = \det\begin{pmatrix} -z & 0 & -1 \\ -2 & 1-z & -1 \\ 1 & 0 & 2-z \end{pmatrix}$$

$$= -z(1-z)(2-z) - (z-1)$$

$$= z(z-1)(2-z) - (z-1)$$

$$= (z-1)\left[z(2-z) - 1\right]$$

$$-z^2 + 2z - 1$$

$$z^2 - 2z + 1$$

$$= -(z-1)(z^2 - 2z + 1) = -(z-1)^3$$

(b) The minimal polynomial $p_{min}(z)$ is either $z-1$, $(z-1)^2$ or $(z-1)^3$. Clearly $A - I \neq 0$. We compute:

$$(A-I)(A-I) =$$

$$\begin{pmatrix} -1 & 0 & -1 \\ -2 & 0 & -1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & -1 \\ -2 & 0 & -1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence $p_{min}(z) = (z-1)^3$

(c) Eigenvalue $\lambda = 1$ must therefore have a Jordan block of size 3, so

$$\boxed{J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}$$

(d) Determine S s.t. $S^{-1} A S = J$

Define $N = A - I$. We know that $N^2 \neq 0$ but $N^3 = 0$. We construct a Jordan chain of length 3. Choose

$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Then $N^2 u = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$Nu = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$. The required Jordan chain is then $\{ N^2 u, Nu, u \}$ and this yields

$$\boxed{S = \begin{pmatrix} 0 & -1 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & 0 \end{pmatrix}}.$$